

# Is the $2D$ $O(3)$ Nonlinear $\sigma$ Model Asymptotically Free?

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## Abstract

We report the results of a Monte Carlo study of the continuum limit of the two dimensional  $O(3)$  non-linear  $\sigma$  model. The notable finding is that it agrees very well with both the prediction inspired by Zamolodchikovs'  $S$ -matrix ansatz and with the continuum limit of the dodecahedron spin model. The latter finding renders the existence of asymptotic freedom in the  $O(3)$  model rather unlikely.

Asymptotic freedom is one of the corner stones of modern particle physics. This special property of certain non-Abelian models is supposed to explain many remarkable properties of quantum field theory, such as the existence of a non-trivial continuum limit, the possibility of grand unification of all interactions, etc. This property was discovered in 1973 in perturbation theory (PT) [1, 2]. However the correctness of this technique within a non-perturbative setting, such as the one offered by lattice quantum field theory, remains mathematically unsettled. In the past, we have repeatedly pointed

out that there were good reasons to doubt the correctness of the PT approach in non-Abelian models [3, 4, 5].

In this letter we would like to report some new numerical results which, in our opinion, make the existence of asymptotic freedom in the two dimensional ( $2D$ ) non-linear  $O(3)$   $\sigma$  model extremely unlikely. Namely we find that the massive continuum limit of this model is, as far as we can see, identical to that of the dodecahedron spin model, which as a discrete spin model must undergo a transition to a phase with long range order (*lro*) at a finite inverse temperature  $\beta$  and hence it cannot possibly be asymptotically free. The only escape out of this bind would be the following scenario: the presently observed agreement between the  $O(3)$  and the dodecahedron spin models is accidental and the phase transition in the dodecahedron model is actually weakly first order, a possibility which we cannot rule out, yet for which there is no theoretical basis.

The  $O(3)$  model is believed to possess only one phase, with exponentially decaying correlations. It is of interest because besides being believed to be asymptotically free, it is a model for which several theoretical predictions have been made. They stem from a conjecture put forward by Zamolodchikov and Zamolodchikov [6] regarding its  $S$ -matrix. Two predictions were made using this  $S$ -matrix ansatz:

- Hasenfratz, Maggiore and Niedermayer (HMN) [7] used it in conjunction with the thermodynamic Bethe ansatz to predict the ratio  $\Lambda/m$
- Balog, using the ideas of Karowski and Weisz [8] about constructing form factors out of the Zamolodchikov  $S$ -matrix [6], extended the work of Kirillov and Smirnov [9] and computed the current 2-point function [10] and then later, together with Niedermaier and Hauer [11], also the 2-point function of the energy-momentum tensor as functions of  $p/m$ . Balog and Niedermaier [12] extended this construction to the two-point functions of the spin and the topological density.

While it has been known for several years that the HMN prediction for  $\Lambda/m$  disagreed with the Monte Carlo value by at least 15%, as far as we know no previous work exists comparing the predictions for the current and spin 2-point functions with Monte Carlo data. In this letter we report the results of some Monte Carlo investigations of these issues. For  $p/m$  less than about 20 we find excellent agreement with the theoretical predictions for the current and spin 2-point functions. On the other hand, even though we reach correlation length  $\xi = 167$  the value of  $\Lambda/m$  disagrees with the HMN prediction by 15% and displays an increase with  $\xi$ , suggesting that it will not

converge to the predicted value.

The numerics were performed using a variant of Wolff's one cluster algorithm [13]. One run consisted of at least 100,000 cluster updates used for thermalization, followed by at least 7 bins of 100,000 clusters used for taking measurements. Thus each run consisted of up to 1,000,000 clusters used for measurements. This procedure was repeated, starting with a different initial configuration; the total number of measurement clusters is recorded in Tab.1. The error was estimated in each run from the bins. If more than one run was performed, we also estimated an error by treating each run as a bin. Typically the latter error was larger and was recorded as the actual error estimate.

The quantities measured were:

- Spin 2-point function  $G(p)$ :

$$G(p) = \frac{1}{L^2} \langle |\hat{s}(p)|^2 \rangle; \quad \hat{s}(p) = \sum_x e^{ipx} s(x) \quad (1)$$

- Current 2-point function  $J(p)$ :

$$J(p) = \frac{1}{L^2} \sum_{a=1}^3 \sum_{\mu=1}^2 \langle |\hat{j}_\mu^a(p)|^2 \rangle; \quad \hat{j}_\mu^a(p) = \sum_x e^{ipx} j_\mu^a(x) \quad (2)$$

where  $j_\mu^a(x) = \beta \epsilon_{abc} s_b(x) s_c(x + \hat{\mu})$

- Magnetic susceptibility  $\chi$ :

$$\chi = G(0) \quad (3)$$

- Correlation length  $\xi$ :

$$\xi = \frac{1}{2 \sin(\pi/L)} \sqrt{(G(0)/G(1) - 1)} \quad (4)$$

We measured these observables in both the  $O(3)$  and dodecahedron spin model. However only in the  $O(3)$  model there exists a continuous symmetry, leading to a conservation law and to Ward identities [14]. The normalization of the current in eq.(2) was fixed by the latter. In the dodecahedron spin model there is no a priori conserved current nor any Ward identities and it is a remarkable finding of our investigation that in the continuum limit the current 2-point function in the dodecahedron spin model is identical, up to an overall normalization, with that in the  $O(3)$  model.

Let us briefly review what is known about the dodecahedron spin model with standard nearest neighbour interaction:

- a) It possesses a high temperature phase with exponential decay of correlation functions.
- b) It possesses a low temperature phase with *lro* and at least one pure phase for each dodecahedron vertex.
- c) It seems to have an intermediate phase with algebraic decay of correlation functions.

While the first two properties follow easily from convergent high and low temperature expansions, the last one has not been proved rigorously. It would follow from a rigorous inequality derived by Richard and us [15], connecting the dodecahedron model with the  $Z(10)$  model, provided one knew that

$$\beta_m(D) > \beta_c(Z(10)). \quad (5)$$

Here  $\beta_m(D)$  is the inverse temperature below which the dodecahedron model exhibits *lro* and  $\beta_c(Z(10))$  the inverse temperature above which  $Z(10)$  exhibits exponential decay. In ref. [16] we gave numerical evidence for the existence of such an intermediate massless phase. In such a phase, one could expect the discrete icosahedral symmetry to be enhanced to  $O(3)$ , just as  $Z(N)$  for  $N > 4$  is enhanced to  $O(2)$  [17]. The results reported in this paper regarding the similarity of the massive continuum limits of the  $O(3)$  and dodecahedron model suggest that indeed this is the case.

We investigated the massive continuum limit in a thermodynamic volume. To be thermodynamic one needs  $L/\xi > 7$  and all of our runs obeyed this criterion. To reach the massive continuum limit one takes  $\xi \rightarrow \infty$  keeping  $p\xi$  fixed. Since numerically one cannot increase  $\xi$  to  $\infty$ , one must control lattice artefacts. We did this in two ways:

1. For a given action, we studied the approach to the continuum limit at fixed  $p/m$  as a function of  $\xi$  ( $m = 1/\xi$ ).
2. We studied the dependence of the results upon the lattice action.

For the standard nearest neighbour action we varied the correlation length from 11 to 167. Besides this action, we took measurements also with the ‘cut-action’ ( $\beta = 0$  but  $\text{grad}(s)$  restricted [18]) and with a ‘Gaussian perfect action’ ( $s(i) \cdot s(j)$  gets replaced by the the square of the angle between the two spins and there is coupling to the nearest neighbour and to the site diagonally located [19]). Our main conclusion is that the continuum limit is approached more slowly at increased  $p/m$  and that for instance at  $p/m=50$

one needs  $\xi$  around 100 for a 1% deviation in  $J(p)$ . We observed no striking effect related to the lattice action used, only perhaps a slight advantage gained with the ‘Gaussian perfect action’ for the dodecahedron spin model (for which the Gaussian improvement is totally ad hoc).

To motivate the presentation of our results let us briefly discuss the form-factor approach of Balog et al. In the spin or current 2-point function they insert the complete set of  $n$ -particle states and write them as an infinite sum, given in terms of form-factors.  $O(N)$  invariance dictates that in  $G(p)$  only odd terms contribute, while in  $J(p)$  only even ones do. Thus  $G(p)$  starts with the 1-particle contribution, which does not depend on the  $S$ -matrix. Thus, to see the dynamics, it is better to subtract this contribution; so we report

$$G_s(x) = [G(x)/G(0) - 1/(x^2 + 1)/1.001687] \times x^2 \quad (6)$$

Here  $x = p/m$ , the division by  $G(0)$  represents the wave function renormalization necessary to have a nontrivial continuum limit and the numerical factor 1.001687 represents the difference between our and Balog’s normalization of the spin 2-point function [20]. The whole expression was multiplied by  $(p/m)^2$  because, according to renormalized perturbation theory, after such multiplication, the expression should diverge as  $\log(p/m)$ . The current 2-point function requires no renormalization (in  $O(3)$ ) and, if one accepts renormalized perturbation theory, is expected to diverge as  $\log(p/m)$  at large  $p$  [10]. In the dodecahedron spin model, the normalization of the current is arbitrary, hence we could either consider  $J(p/m)/J(1)$  and compare it with the same quantity measured in  $O(3)$  or simply see if after multiplication by a constant  $J(p)$  in the two models coincide.

#### *Results:*

Perturbative renormalization group arguments lead to the predictions

$$\xi \sim \frac{C_\xi e^{2\pi\beta}}{2\pi\beta} (1 + \sum_{k \geq 1} \frac{a_k}{\beta^k}); \quad \chi \sim \frac{C_\chi e^{4\pi\beta}}{(2\pi\beta)^4} (1 + \sum_{k \geq 1} \frac{b_k}{\beta^k}) \quad (7)$$

the action dependent coefficients  $a_k$  and  $b_k$  are known for the standard action up to 4 loops [22]. The HMN prediction is that

$$C_\xi = \frac{e}{8} 2^{-5/2} e^{-\pi/2} \approx 0.012487 \quad (8)$$

According to ref.[23, 21] the HMN prediction implies also  $C_\chi = 0.0145$ .

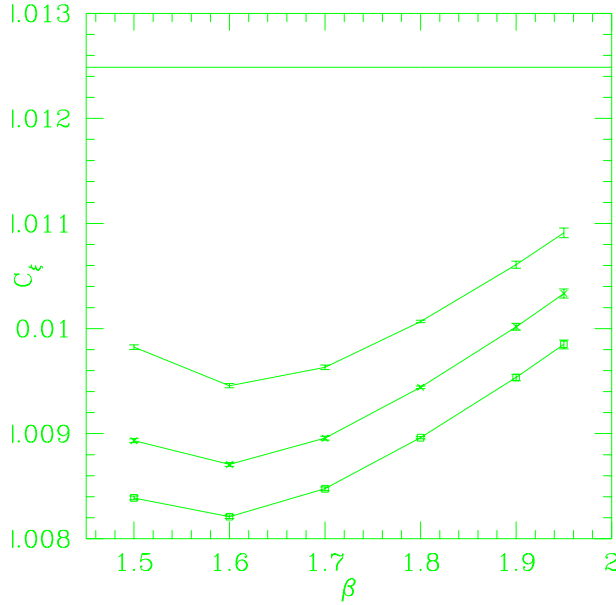


Figure 1:  $C_\xi = \Lambda/m$  compared to the HMN prediction (horizontal line). The data points correspond upwards to 2 loop, 3 loop and 4 loop PT, respectively.

In Fig.1 we present our determination of  $\Lambda/m$  in the  $O(3)$  model with standard action, in Fig.2 our determination of  $C_\chi$ , compared to the predictions by [7] and [23], respectively. The values measured and the number of clusters used are recorded in Tab.1.

In Fig.4 and 3) we compare  $G_s(p/m)$  (eq.(6)) and  $J(p/m)$  (eq.(2)), respectively, for the  $O(3)$  model with standard action at different correlation lengths with the prediction of Balog and Niedermaier [12] and Balog [10], respectively. The errors are less than  $10^{-3}$  and therefore not shown.

**Tab.1:** Monte Carlo data for the standard action

$\beta$	$L$	clusters	$\xi$	$\chi$
1.5	78	$1 \times 10^6$	11.03(2)	175.75(43)
1.6	140	$1 \times 10^6$	18.97(4)	447.30(6)
1.7	250	$2 \times 10^6$	34.55(7)	1268.(3.)
1.8	500	$4 \times 10^6$	64.67(8)	3831.(6.)
1.9	910	$2 \times 10^6$	122.15(39)	11847.(67.)
1.95	1200	$7 \times 10^5$	168.39(71)	21148.(123.)

In Fig.5 (6) we compare  $G_s(p/m)$  ( $J(p/m)$ ) for the  $O(3)$  and the dodec-

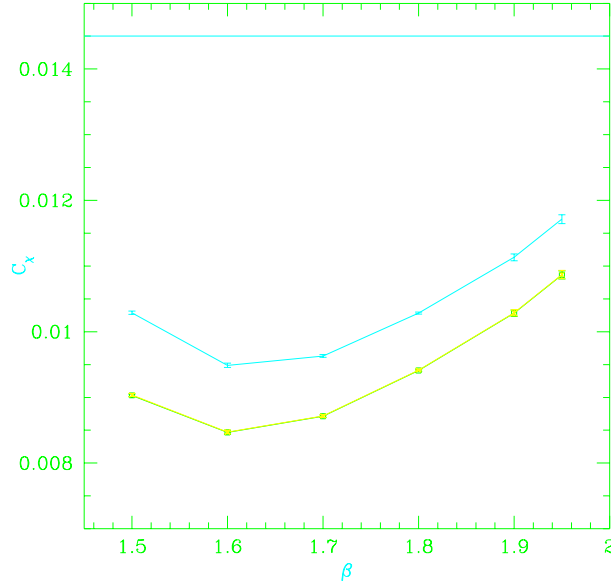


Figure 2:  $C_\chi$  compared to the prediction of Alles et al (horizontal line). The data points correspond upwards to 2 loop and 3 loop (indistinguishable) and 4 loop PT, respectively.

ahedron models, both with standard action. For the dodecahedron model  $J(p/m)$  was multiplied by a renormalization factor determined by fitting.

*Discussion:*

1. The data in Fig.1 agree very well with prior data [24, 25] and indicate that most likely  $\Lambda/m$  will increase past the HMN prediction. Attempts to improve this situation by using the so-called ‘energy scheme’ [23] produced similar results (the curve is shifted closer to the predicted value but it shows a non-vanishing slope away from it).

2. A comparison of  $\chi(\beta)$  with the predictions of asymptotic freedom shows the same behavior: the data increase faster than expected, in agreement with the existence of a critical point at finite  $\beta$ . From the data in Tab.1 we find that  $C_\chi$  is growing steadily, so far reaching 0.0111, compared to the ‘exact’ prediction of a constant value of 0.0145.

3. The numerical agreement of  $G_s(p/m)$  and (the normalization adjusted)  $J(p/m)$  in the  $O(3)$  and dodecahedron spin model and their excellent agreement with the Balog and Balog-Niedermaier prediction strongly suggest that the two models possess the same continuum limit, which is correctly pre-

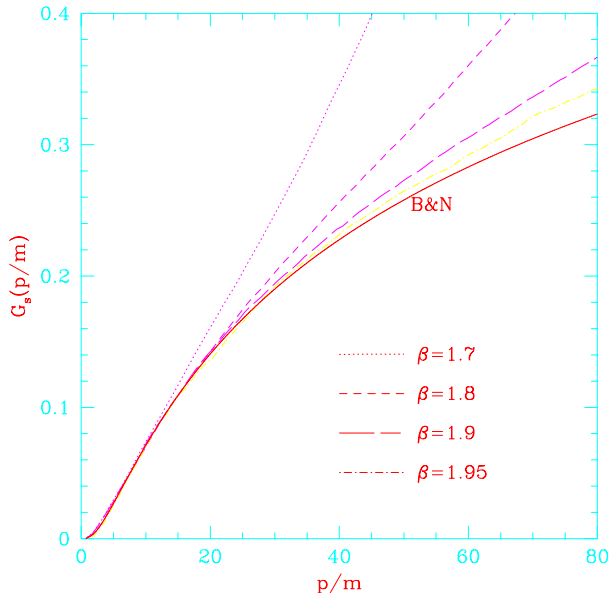


Figure 3:  $G_s(p)$  for  $O(3)$  as a function of  $p/m$ . The solid line is the prediction of Balog and Niedermaier.

dicted by the Zamolodchikovs  $S$ -matrix ansatz. Since the massive phase of the dodecahedron model terminates at finite  $\beta$ , its continuum limit cannot have asymptotic freedom. Indeed, for instance the running coupling  $g_r(L)$  introduced in [26] cannot vanish at short distances, because at any  $\beta$ : (a)  $g_r(L)_{O(2)}$  does not vanish and (b)  $g_r(L)_{O(3)} \geq g_r(L)_{O(2)}$  [27].

4. Balog and Niedermaier[28] have put forth an ansatz for the contribution of higher  $n$ -particles to  $G(p/m)$  and  $J(p/m)$ . According to them this ansatz would explain why the contribution of  $n > 6$  is negligible at  $p/m < 10^4$  and how the same contributions sum up at asymptotic  $p/m$  to produce the logarithmic increase of both  $G_s(p/m)$  and  $J(p/m)$  corresponding to asymptotic freedom. Our data suggest that their ansatz is inadequate, since the agreement of  $O(3)$  with the dodecahedron model rules out such a logarithmic increase in  $J(p)$  (see [14]).

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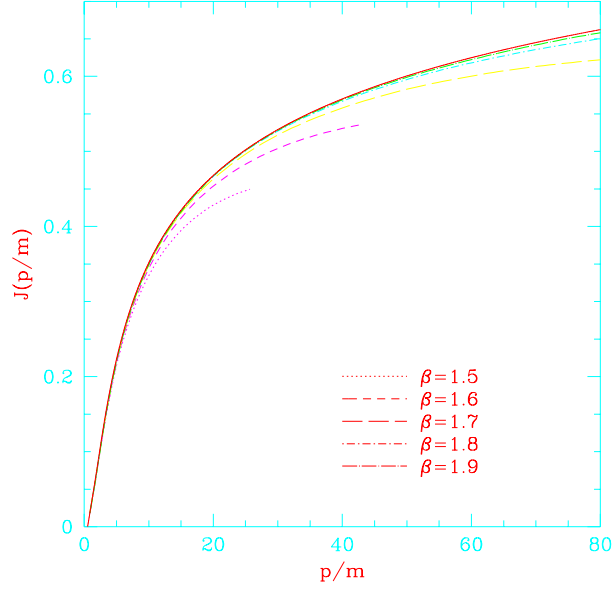


Figure 4:  $J(p)$  for  $O(3)$  as a function of  $p/m$ . The solid line is the prediction of Balog and Niedermaier.

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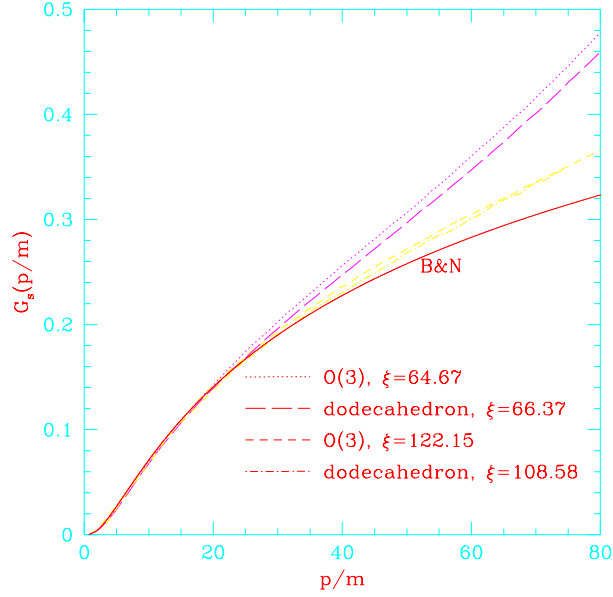


Figure 5:  $G_s(p)$  for  $O(3)$  and the dodecahedron as a function of  $p/m$ . The solid line is the prediction of Balog and Niedermaier.

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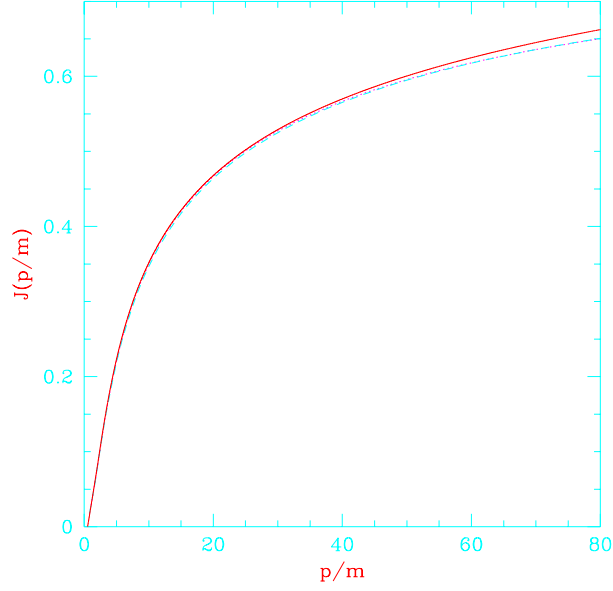


Figure 6:  $J(p)$  for  $O(3)$  (dotted) and the dodecahedron (dashed) as a function of  $p/m$ . The solid line is Balog's prediction.

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